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LETTER TO THE EDITOR

A new type of pseudoparticle solution to the Yang–Mills equation

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Abstract. By making use of the method of the phase factor of the standard differential loop, the more general forms of the imbedding SO(4) pseudoparticle solutions with vanishing energy–momentum tensor in SU(N) Yang–Mills theory are obtained.

The remarkable BPST pseudoparticle solutions to SU(2) Yang–Mills (YM) theory in the four-dimensional Euclidean space have attracted much attention since their discovery (Belavin *et al* 1975). Many possible physical effects of these solutions have also been suggested. The solutions seem to be connected with the anomalous divergence of U(1) axial vector current and the chiral symmetry breakdown—several of the deep problems in particle physics (Pagels 1976, Weinberg 1975). Deser showed that the pseudoparticle solutions with finite energy exist only in the four-dimensional Euclidean space (Deser 1976). Jackiw and Rebbi (1976) discussed the conformal properties of BPST solutions and pointed out that they are invariant under the SO(5) subgroup of conformal transformations. Wu *et al* (1978) gave a united expression of the solutions given by BPST and Jackiw and Rebbi. In this paper we derive the more general forms of the imbedding SO(4) pseudoparticle solutions in SU(N) YM gauge theory by making use of the method of the phase factor of the standard differential loop (PFSDL) (Gu 1981, Ma 1982, 1984). The solutions obtained have vanishing energy–momentum tensors, and hence represent vacuum solutions. Generally, they are not self-dual or anti-self-dual. Some explicit expressions for pseudoparticle solutions of SU(3), SU(4) and SU(5) gauge theories are discussed.

In Euclidean space, $g_{\mu\nu}=\delta_{\mu\nu}$, no distinction need be made between upper and lower indices. It is reasonable to believe that the solution in the ground state should possess larger symmetry, so we look for the spherically symmetric pseudoparticle solutions in the four-dimensional Euclidean space, namely, the gauge potentials which are gauge equivalent to each other under four-rotations. For definiteness we discuss the SU(N) gauge field.

Following the method of PFSDL, we define the standard path for any point P as the straight line from the origin O to P. The standard differential loop $OP(P+\mathrm{d}P)O$ is composed of the differential path $P(P+\mathrm{d}P)$ and two neighbouring standard paths OP and $(P+\mathrm{d}P)O$. The Lie algebraic part k_μ of the phase factor $\Phi_{O(P+\mathrm{d}P)PO}$ of this

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loop is called the phase factor of the standard differential loop:

$$\Phi_{O(P+\mathrm{d}P)PO} = \mathbb{1} - \mathrm{i}ek_\mu(x)\,\mathrm{d}x_\mu. \tag{1}$$

From the definition, k_μ has the following properties.

(i) The consistency conditions

$$k_\mu(0) = 0, \qquad k_\mu(x)x_\mu = 0. \tag{2}$$

(ii) Under the local gauge transformations $U(x)$, it transforms through the constant similarity transformation

$$k'_\mu = uk_\mu u^{-1}, \qquad u = U(0). \tag{3}$$

(iii) The necessary and sufficient condition for the equivalency of two potentials is the existence of a definite matrix $u \in \mathrm{SU}(N)$ such that their PFSDL k_μ satisfy equation (3).

(iv) Each PFSDL k_μ can determine an equivalent class of gauge potentials and k_μ itself is a member of this class. Consequently, we can choose a useful gauge called the central gauge so that the gauge potential W_μ itself is PFSDL. In the following discussions, we restrict ourselves to the central gauge.

From the definition of spherically symmetric gauge potential and the property of (iii), the gauge potential transforms under four-rotations as

$$R_{\mu\nu}W_\nu(R^{-1}x) = \mathscr{D}^{-1}(R)\,W_\mu(x)\mathscr{D}(R), \tag{4}$$

where $R_{\mu\nu}$ is a four-rotation matrix, and $\mathscr{D}(R)$ is an N-dimensional representation of the SO(4) group. Equation (4) implies that a four-rotation can be compensated by a global gauge transformation, that is, the spherically symmetric gauge potential which is regular in the whole space must be synchrospherically symmetric. Notice that the generators $I_{\mu\nu}$ of $\mathscr{D}(R)$ satisfy the following equation

$$\mathscr{D}(R)^{-1}I_{\mu\nu}\mathscr{D}(R) = R_{\mu\rho}R_{\nu\lambda}I_{\rho\lambda}. \tag{5}$$

The general form of synchrospherically symmetric gauge potential composed of $I_{\mu\nu}$ and x_μ can be expressed as

$$W_\mu(x) = \phi_1(x^2)I_{\mu\nu}x_\nu + \phi_2(x^2)\tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}I_{\nu\rho}x_\lambda. \tag{6}$$

Evidently, equation (6) satisfies equations (2) and (4). Since $I_{\mu\nu}$ belongs to SO(4) subalgebra, the gauge potentials W_μ we obtain are only the imbedding solutions of the SO(4) gauge field. From equation (6), we can calculate the gauge field strength $G_{\mu\nu}$ in a straightforward way

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu - \mathrm{i}e[W_\mu, W_\nu] \\ &= [-2\phi_1 + ex^2(\phi_1^2+\phi_2^2)]I_{\mu\nu} - \phi_2\varepsilon_{\mu\nu\rho\lambda}I_{\rho\lambda} \\ &\quad + [2\dot{\phi}_1 + e(\phi_1^2+\phi_2^2)](x_\mu I_{\nu\rho} - x_\nu I_{\mu\rho})x_\rho \\ &\quad + \dot{\phi}_2(x_\mu\varepsilon_{\nu\sigma\rho\lambda} - x_\nu\varepsilon_{\mu\sigma\rho\lambda})I_{\sigma\rho}x_\lambda - 2e\phi_1\phi_2\varepsilon_{\mu\nu\sigma\lambda}x_\rho I_{\rho\sigma}x_\lambda \end{aligned} \tag{7}$$

where the dots refer to differentiation with respect to x^2. Inserting equations (6) and (7) into the sourceless YM equation

$$D_\mu G_{\mu\nu} = \partial_\mu G_{\mu\nu} - \mathrm{i}e[W_\mu, G_{\mu\nu}] = 0, \tag{8}$$

we have

$$x^2(2\ddot{\phi}_1 - e^2\phi_1^3 - 3e^2\phi_1\phi_2^2) + 3(2\dot{\phi}_1 + e\phi_1^2 + e\phi_2^2) = 0, \tag{9a}$$

$$x^2(2\ddot{\phi}_2 - e^2\phi_2^3 - 3e^2\phi_1^2\phi_2) + 6(\dot{\phi}_2 + e\phi_1\phi_2) = 0. \tag{9b}$$

Adding and subtracting (9a) and (9b), we can obtain the special solutions of the form

$$\begin{aligned} &\phi_1 + \phi_2 = \frac{2}{e(x^2+a^2)}, && \phi_1 - \phi_2 = \frac{2}{e(x^2+b^2)}, \\ &\phi_1 = \frac{1}{e}\left[\frac{1}{x^2+a^2} + \frac{1}{x^2+b^2}\right], && \phi_2 = \frac{1}{e}\left[\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right], \end{aligned} \tag{10}$$

where we choose the integral constants $a^2 > 0$, $b^2 > 0$, so that ϕ_1, ϕ_2 and hence W_μ are regular in the whole space. Now, we substitute (10) into (7) with the aid of the identity

$$\tfrac{1}{2}(x_\mu \varepsilon_{\nu\sigma\rho\lambda} - x_\nu \varepsilon_{\mu\sigma\rho\lambda}) I_{\sigma\rho} x_\lambda + \varepsilon_{\mu\nu\sigma\lambda} x_\rho I_{\rho\sigma} x_\lambda = -\tfrac{1}{2} x^2 \varepsilon_{\mu\nu\sigma\rho} I_{\sigma\rho}; \tag{11}$$

then $G_{\mu\nu}$ reduces to

$$\begin{aligned} &G_{\mu\nu} = A I_{\mu\nu} + \tfrac{1}{2} B \varepsilon_{\mu\nu\sigma\rho} I_{\sigma\rho}, \\ &A = -2\phi_1 + ex^2(\phi_1^2 + \phi_2^2) = -\frac{2}{e}\left[\frac{a^2}{(x^2+a^2)^2} + \frac{b^2}{(x^2+b^2)^2}\right], \\ &B = -2\phi_2 + 2ex^2\phi_1\phi_2 = -\frac{2}{e}\left[\frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+b^2)^2}\right]. \end{aligned} \tag{12}$$

Since $SO(4) \sim SU(2) \times SU(2)$, we can introduce, instead of $I_{\mu\nu}$, the following two sets of SU(2) generators L_a, K_a $(a = 1, 2, 3)$

$$\begin{aligned} &I_{ab} = \varepsilon_{abc}(L_c + K_c), && I_{a4} = L_a - K_a, \\ &[L_a, L_b] = \mathrm{i}\varepsilon_{abc} L_c, && [K_a, K_b] = \mathrm{i}\varepsilon_{abc} K_c, \\ &[L_a, K_b] = 0. \end{aligned} \tag{13}$$

For the irreducible representations $\mathscr{D}^{jk}(R) = \mathscr{D}^{j0}(R) \times \mathscr{D}^{0k}(R)$, we have

$$L_a^{jk} = I_a^j \times \mathbb{1}_{2k+1}, \qquad K_a^{jk} = \mathbb{1}_{2j+1} \times I_a^k, \tag{14}$$

where I_a^j denotes the generator of the representation D^j of SU(2), and $\mathbb{1}_n$ is the $n \times n$ unit matrix. $\mathscr{D}(R)$ which belongs to SU(N) is the representation of the group SO(4), and can be reduced into the direct sum of the irreducible representations $\mathscr{D}^{jk}(R)$ of SO(4).

From equation (12) it follows that

$$\begin{aligned} &\mathrm{Tr}(G_{\mu\sigma} G_{\nu\sigma}) = C_1 \delta_{\mu\nu}(A+B)^2 + C_2 \delta_{\mu\nu}(A-B)^2, \\ &\mathrm{Tr}({}^*G_{\mu\sigma} G_{\nu\sigma}) = C_1 \delta_{\mu\nu}(A+B)^2 - C_2 \delta_{\mu\nu}(A-B)^2, \end{aligned} \tag{15}$$

where C_1, C_2 are the Casimir operators of SO(4)

$$C_1 = \mathrm{Tr}(\boldsymbol{L}^2), \qquad C_2 = \mathrm{Tr}(\boldsymbol{K}^2), \tag{16}$$

and ${}^*G_{\mu\nu}$ is the dual tensor of $G_{\mu\nu}$. In terms of equation (15) it is easy to prove that the energy–momentum tensor of the solutions vanishes,

$$\theta_{\mu\nu} = \tfrac{1}{2}\mathrm{Tr}[-\delta_{\mu\nu}G_{\rho\lambda}G_{\rho\lambda} + 4G_{\mu\rho}G_{\nu\rho}] = 0. \tag{17}$$

It implies that the pseudoparticle solutions of equation (12) are vacuum solutions. Generally, these solutions are not self-dual or anti-self-dual. In the special cases of $\mathscr{D}^{jk} = \mathscr{D}^{j0}$ or $\mathscr{D}^{0k}$, both terms of equation (12) are self-dual or anti-self-dual, respectively. The well known SU(2) pseudoparticle solutions belong to these cases. The other special cases are $b \to \infty$ ($\phi_1 = \phi_2$, $A = B$, self-dual) or $a \to \infty$ ($\phi_1 = -\phi_2$, $A = -B$, anti-self-dual).

We now discuss the solutions in some simple gauge fields.

(i) *SU(2) gauge field*

$\mathscr{D}$ can be the representations of $\mathscr{D}^{\frac{1}{2}0}$ or $\mathscr{D}^{0\frac{1}{2}}$,

$$\mathscr{D}^{\frac{1}{2}0}: \qquad \boldsymbol{W} = \tfrac{1}{2}(\phi_1 + \phi_2)[\boldsymbol{r} \wedge \boldsymbol{\sigma} + \boldsymbol{\sigma}x_4], \qquad W_4 = -\tfrac{1}{2}(\phi_1 + \phi_2)\boldsymbol{\sigma} \cdot \boldsymbol{r}, \tag{18a}$$

$$\mathscr{D}^{0\frac{1}{2}}: \qquad \boldsymbol{W} = \tfrac{1}{2}(\phi_1 - \phi_2)[\boldsymbol{r} \wedge \boldsymbol{\sigma} - \boldsymbol{\sigma}x_4], \qquad W_4 = \tfrac{1}{2}(\phi_1 - \phi_2)\boldsymbol{\sigma} \cdot \boldsymbol{r}, \tag{18b}$$

where $\boldsymbol{r} = (x_1, x_2, x_3)$. Equation (18*a*) is the form given by BPST, and (18*b*) is given by Jackiw and Rebbi (1976).

(ii) *SU(3) gauge field*

Besides $\mathscr{D}^{\frac{1}{2}0} \oplus \mathscr{D}^{00}$ and $\mathscr{D}^{0\frac{1}{2}} \oplus \mathscr{D}^{00}$, $\mathscr{D}$ can take the representations of $\mathscr{D}^{10}$ and $\mathscr{D}^{01}$,

$$\mathscr{D}^{10}: \qquad \boldsymbol{W} = (\phi_1 + \phi_2)[\boldsymbol{r} \wedge \boldsymbol{T} + \boldsymbol{T}x_4], \qquad W_4 = -(\phi_1 + \phi_2)\boldsymbol{T} \cdot \boldsymbol{r}, \tag{19a}$$

$$\mathscr{D}^{01}: \qquad \boldsymbol{W} = (\phi_1 - \phi_2)[\boldsymbol{r} \wedge \boldsymbol{T} - \boldsymbol{T}x_4], \qquad W_4 = (\phi_1 - \phi_2)\boldsymbol{T} \cdot \boldsymbol{r}, \tag{19b}$$

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(iii) *SU(4) gauge field*

For $\mathscr{D}^{\frac{1}{2}0} \oplus 2\mathscr{D}^{00}$ or $\mathscr{D}^{0\frac{1}{2}} \oplus 2\mathscr{D}^{00}$, the solution has the form of equation (18). For $\mathscr{D}^{10} \oplus \mathscr{D}^{00}$ or $\mathscr{D}^{01} \oplus \mathscr{D}^{00}$, it is like the solution of equation (19). $\mathscr{D} = \mathscr{D}^{\frac{1}{2}0} \oplus \mathscr{D}^{0\frac{1}{2}}$ corresponds to the solution given by Jackiw and Rebbi (1976). When $\mathscr{D} = \mathscr{D}^{\frac{1}{2}\frac{1}{2}}$, which is a real representation, the solution is also one of SO(4) gauge field just like that given by BPST ($\phi_2 = 0$). This solution was rewritten in terms of $I_{\mu\nu}x_\nu$ by Wu *et al* (1978). The general form should include the terms ϕ_2 as follows

$$\begin{aligned} \boldsymbol{W} &= (\phi_1 + \phi_2)[\boldsymbol{r} \wedge \boldsymbol{L} + \boldsymbol{L}x_4] + (\phi_1 - \phi_2)[\boldsymbol{r} \wedge \boldsymbol{K} - \boldsymbol{K}x_4], \\ W_4 &= -(\phi_1 + \phi_2)\boldsymbol{L} \cdot \boldsymbol{r} + (\phi_1 - \phi_2)\boldsymbol{K} \cdot \boldsymbol{r}. \end{aligned} \tag{20}$$

Besides, $\mathscr{D}$ can take $\mathscr{D}^{\frac{3}{2}0}$ or $\mathscr{D}^{0\frac{3}{2}}$, and the solution can be obtained after substituting the corresponding generators (14) into equation (20).

(iv) *SU(5) gauge field*

$\mathscr{D}$ can be $\mathscr{D}^{\frac{1}{2}0} \oplus 3\mathscr{D}^{00}$, $\mathscr{D}^{0\frac{1}{2}} \oplus 3\mathscr{D}^{00}$, $2\mathscr{D}^{\frac{1}{2}0} \oplus \mathscr{D}^{00}$, $2\mathscr{D}^{0\frac{1}{2}} \oplus \mathscr{D}^{00}$, $\mathscr{D}^{\frac{1}{2}0} \oplus \mathscr{D}^{0\frac{1}{2}} \oplus \mathscr{D}^{00}$, $\mathscr{D}^{10} \oplus 2\mathscr{D}^{00}$, $\mathscr{D}^{01} \oplus 2\mathscr{D}^{00}$, $\mathscr{D}^{01}$, $\mathscr{D}^{10} \oplus \mathscr{D}^{0\frac{1}{2}}$, $\mathscr{D}^{01} \oplus \mathscr{D}^{\frac{1}{2}0}$, $\mathscr{D}^{01} \oplus \mathscr{D}^{0\frac{1}{2}}$, $\mathscr{D}^{\frac{1}{2}\frac{1}{2}} \oplus \mathscr{D}^{00}$, $\mathscr{D}^{\frac{3}{2}0} \oplus \mathscr{D}^{00}$, $\mathscr{D}^{0\frac{3}{2}} \oplus \mathscr{D}^{00}$, $\mathscr{D}^{20}$ and $\mathscr{D}^{02}$. Substituting equation (14) into equation (20), we can obtain the corresponding forms of the pseudoparticle solutions. They are all the SO(4) imbedding solutions.

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